Meson, Baryons, and Other Large N Animals

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Abstract

In the following review, we give an introduction to 't Hooft's [9] $1/N$ expansion. After presenting the main ideas and machinery, we focus on its application to QCD, including some aspects of meson and baryon phenomenology.

1 Introduction

By far the most powerful tool in our calculational arsenal is perturbation theory. As long as we have a Lagrangian and a small coupling, we can determine decay widths and scattering cross sections to any desired level of accuracy. When no small expansion parameter exists, we have little recourse for accessing interesting, strongly coupled physics. QCD, for example, features such important phenomena as chiral symmetry-breaking and quark confinement at strong coupling.

The large N expansion is based on a simple but beautiful idea due to 't Hooft: even if an $SU(N)$ or $SO(N)$ gauge theory has no small coupling, it is possible to expand in $1/N$ as $N \to \infty$. The order in $1/N$ of each Feynman diagram is determined by a topological invariant called the Euler characteristic. This provides a simple tool for determining the relative amplitudes of scattering processes.

While we have other very sophisticated tools to investigate the strongly coupled regime of QCD (such as lattice gauge theory), the large N expansion provides a universally applicable method for understanding $SU(N)$ gauge theories – or any theory with some $SU(N)$ symmetry group. Furthermore, in high energy phenomenology, it successfully reproduces such low energy

results as the Zweig rule and suppression of exotics, even though $N = 3$ for QCD – which hardly seems "large!"

The large N expansion also has interesting applications in statistical physics. For example, in systems of ferromagnets close to the critical temperature for a transition between ordered and disordered phases, the Hamiltonian of the model often exhibits an $SO(3)$ symmetry. Generalizing the system to d spatial dimensions for large d, and expanding correlators in powers of $1/d$ allows us to examine regimes where the Hamiltonian otherwise lacks a small expansion parameter[13].

More recently, the large N expansion has played a key role in the AdS/CFT correspondence. The perturbative expansion of string theory, like the large N expansion, is given in terms of the string coupling to the power of the worldsheet's Euler characteristic. Making the connection to the large N expansion, we see that the closed string coupling is proportional to $1/N$: as $N \to \infty$ the string coupling is small and we have a weakly coupled string theory. The spectrum of large N closed strings is the glueball spectrum (to leading order in $1/N$) and the open string spectrum produces mesons, so we might be able to use string methods to study the non-perturbative regime of QCD. For instance, string theory could show that the meson and glueball spectra are discrete, which would give definitive proof of confinement [6]. The AdS/CFT correspondence gives us a tool for doing this. It is conjectured [3] that weakly coupled superstring theory (i.e. supergravity) in the ten-dimensional space $AdS_5 \times S^5$ is dual to a strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory living on the boundary of the AdS_5, for reasons outlined above.[1] Though work on the correspondence is still in its preliminary stages (and no formal proof exists for its accuracy), we may eventually be able to use supergravity calculations in the bulk to learn about the strong coupling limit of QCD.

2 Machinery

2.1 Large N in ϕ^4

Before confronting the more complicated case of QCD, we begin with an $O(N)$-symmetric ϕ^4 theory [2], where counting factors of N's is more straight-

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[1] AdS_5, or five-dimensional Anti-de Sitter space, is a solution to Einstein's equations with constant negative curvature.
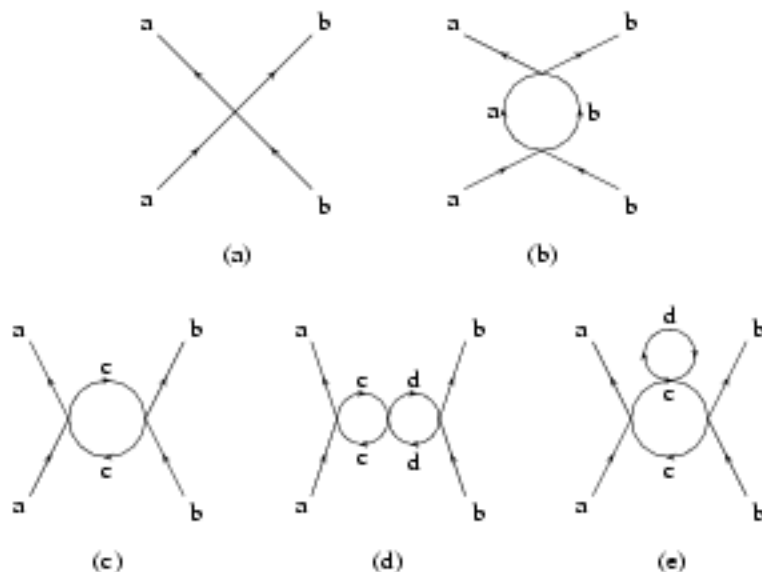
Figure 1: Diagrams for $\phi^a\phi^a \to \phi^b\phi^b$ scattering at various orders in N and λ.

forward. The field ϕ^a transforms in the fundamental of $O(N)$ ($a = 1 \ldots n$), and has the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a - \frac{1}{8}\frac{\lambda}{\sqrt{N}}(\phi^a\phi^a)^2. \tag{1}$$

As usual, summation over a is implied, and the coupling λ has been rescaled by the factor of $\sqrt{N}$ for later convenience. The ϕ^4 term gives a vertex proportional to $\frac{\lambda}{\sqrt{N}}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$. For the scattering process $\phi^a\phi^a \to \phi^b\phi^b$, the diagrams contributing to leading, one- and two-loop orders are as shown in Fig. 1(a)-(e).[2] Fig. 1(a) is clearly of order $\lambda/\sqrt{N}$. Fig. 1(b) has two vertices and no summation over the $O(N)$-vector index: it is of order λ^2/N. Fig. 1(c) also has two vertices, but the initial and final legs are arranged such that we have a "color" loop. That is, there is a sum over the index c, which gives an additional N, and boosts the order of the diagram

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[2]This figure, and all of the subsequent figures which display standard diagrams found in most reviews and texts on the $1/N$ expansion, are taken from the Les Houches Lectures of A.V. Manohar [4].

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to λ^2. Fig. 1(d) and 1(e) are order $\lambda^3 \sqrt{N}$. We now begin to see that the $1/N$ expansion is orthogonal to the expansion for a small λ : diagrams of equal order in the coupling can have different orders of N. At each order in N, we can find diagrams at essentially all orders in λ. In this sense the $1/N$ expansion at any given order of N is *exact* in the coupling. [3] Since we cannot write down even the entire first term of the $1/N$ expansion in closed form, this statement is not terribly practical, but it shows nonetheless that we have arrived at a "nonperturbative perturbative expansion."

To demonstrate that $1/\sqrt{N}$ is indeed a valid expansion parameter as $N \to \infty$, we must show that there is some upper limit to the order in N of any given diagram. We introduce a nondynamical auxiliary field, σ, to make the calculation more transparent. Now

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}m^2 \phi^a \phi^a + \frac{\sqrt{N}}{2\lambda}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a. \tag{2}$$

Note that inserting the equation of motion for σ ($\sigma = \frac{\lambda}{2\sqrt{N}}\phi^a\phi^a$) into Eq. (2), recovers our original Lagrangian. Now, the only interactions are vertices of one σ and two ϕ's, which are $\mathcal{O}(1)$ in N. The N-dependence of each diagram is now determined entirely by the number of σ propagators, each of order $1/\sqrt{N}$, and the number of ϕ loops, each $\mathcal{O}(N)$. As Coleman [2] suggests, we can simply strip away all external ϕ lines, and integrate out the momenta of the ϕ loops for an effective action entirely in terms of σ. The integrated ϕ loops give some nonlocal interaction among the σ fields. In the usual path integral formalism,

$$e^{iS_{eff}(\sigma)} = \int \prod_a [d\phi^a] e^{iS(\phi^a, \sigma)}. \tag{3}$$

As all terms in the new Lagrangian with σ (Eq. (2)) involve quadratic powers of ϕ^a, with no mixing between ϕ's of different "color" indices, the integral on the right-hand-side of Eq. (3) separates into a path integral over a single ϕ (i.e. $N = 1$) all raised to the Nth power. This implies that

$$S_{eff}(\sigma, N = N') = N' S_{eff}(\sigma, N = 1). \tag{4}$$

We can now characterize the diagrams of the effective σ theory according to their powers of N. Each of E external σ lines adds a factor of $1/\sqrt{N}$, as

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[3]Though the statement is essentially trivial, note that the converse is also true: if we expand for small λ, each term is exact in N.

does each of I internal σ lines. The ϕ integrations that give us the nonlocal σ vertices add a factor of N at each of V vertices. For connected graphs, the number of loops, L, is given by $L = I - V + 1$, which provides an alternative formulation. The order of a given diagram is thus $N^{V-(I+E)/2} = N^{1-L-(E-I)/2}$. The lowest order diagram is one with no loops and the minimal number of external lines. The number of external lines is always greater than the number of internal lines, so the highest possible order in N for any connected diagram is $\sqrt{N}$.[4]

2.2 Counting N's in the Vacuum

To analyze the large N behavior of $SU(N)$ gauge theories, we proceed as for ϕ^4 theory: we write down the Lagrangian rescaled with appropriate factors of N, and determine a generic formula for the order of any given diagram. The complication in the Yang-Mills (YM) case results from the gauge fields, which transform in the adjoint (rather than the fundamental) of the gauge group. For the ϕ^4 theory, we could only make one gauge singlet, $\phi^a \phi^a$, which we associated with the auxiliary field σ. In QCD, tracing over many powers of the field strength $F_{\mu\nu}$ gives a gauge singlet, so that technique for simplifying the Feynman diagrams of ϕ^4 is no longer practical. Instead, we take advantage of 't Hooft's revelation [9] that the topology of a given graph uniquely defines its order in $1/N$.

As we did for λ in the ϕ^4 case, we rescale the coupling from g to $g/\sqrt{N}$ in the usual YM Lagrangian, to find a sensible large N expansion [4]. We are led to this choice by examination of the beta function, which (before the rescaling of g) has the familiar form

$$\beta = \mu \frac{\partial g}{\partial \mu} = -\left(\frac{11}{3} N - \frac{2}{3} N_f \right) \frac{g^3}{16\pi^2} + \mathcal{O}(g^5). \tag{5}$$

For a fixed number N_f of flavors, this β is $\mathcal{O}(N)$, and does not give a well-defined limit as N approaches infinity. Rescaling g as suggested, we find

$$\beta = -\left(\frac{11}{3} - \frac{2}{3} \frac{N_f}{N} \right) \frac{g^3}{16\pi^2} + \mathcal{O}(g^5), \tag{6}$$

which not only has a suitable limit as $N \to \infty$, but also shows that fermion loops will be suppressed as $1/N$ [4].

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[4]The order of a disconnected diagram is simply the product of the orders of its connected pieces.

The Lagrangian for an $SU(N)$ gauge theory with N_f flavors now takes the form

$$\mathcal{L} = -\frac{1}{2}\text{Tr}[F_{\mu\nu}F^{\mu\nu}] + \sum_{i=1}^{N_f} \bar{\psi}_i(i\gamma^\mu D_\mu - m_i)\psi_i. \qquad (7)$$

The covariant derivative and the field strength are defined in terms of the rescaled coupling:

$$D_\mu = \partial_\mu + i\frac{g}{\sqrt{N}}A_\mu \text{ and } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{ig}{\sqrt{N}}[A_\mu, A_\nu], \qquad (8)$$

where the gauge field $A_\mu = A_{\mu C}t_C$.

To implement 't Hooft's trick, it will prove convenient to think of the gauge fields as $N \times N$ matrices $A^a_{\mu b}$, having two color indices, a and b. The index a transforms in the anti-fundamental, and b in the fundamental. In terms of the traceless, hermitian generators t_C, and the conventional gauge field $A_{\mu C}$, we have $A^a_{\mu b} = A_{\mu C}(t_C)^a_b$.

From the Lagrangian, we read off the propagators:

$$\langle \psi^a(x)\bar{\psi}^b(y)\rangle = \delta^{ab}S(x-y) \qquad (9)$$

where $S(x-y)$ is the usual Dirac propagator. From the original gluon propagator $\langle A^C_\mu(x)A^D_\nu(y)\rangle = \delta_{CD}D_{\mu\nu}(x-y)$, the propagator for our new matrix-valued gauge fields [4] is:

$$\langle A^a_{\mu b}(x)A^c_{\nu d}(y)\rangle = D_{\mu\nu}(x-y)\left(\frac{1}{2}\delta^a_d\delta^c_b - \frac{1}{2N}\delta^a_b\delta^c_d\right). \qquad (10)$$

The coefficient of the abelian gauge field propagator $D_{\mu\nu}(x-y)$ comes from the normalization of the $SU(N)$ generators. It is easy to check, for instance, that the second term ensures tracelessness. Each fermion propagator is thus $\mathcal{O}(N^0)$, as is the *leading* order of each gluon propagator[5].

We now turn to the Feynman diagrams. 't Hooft [9] recognized that by writing the gluon fields in the form described above, he could treat them as quark-antiquark pairs. In his "double-line" notation we draw a line for each *index*: quark propagators are one line, and gluon propagators are two

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[5]This means that for most purposes we can consider a $U(N)$ instead of an $SU(N)$ gauge field, as the term inserted to impose tracelessness of the generators goes as $1/N$; its effect will always be suppressed with respect to the leading order of a given process.

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lines, with arrows pointing in opposite directions as appropriate for a quark and an antiquark. This yields diagrams where color "flow" is preserved (we have no vertices with different numbers of quarks flowing in or out), because all interactions come from a single color trace in the Lagrangian. Some important examples of the notation are given in Fig.2. Amputating all external legs and considering only vacuum diagrams for the moment, we can interpret the color loops as a series of polygons glued together to form a two-dimensional surface. The surface is oriented because the direction of the arrows determines a particular orientation for each face[6]. If we consider only amputated (or vacuum-to-vacuum) graphs, each diagram is analogous to a compact manifold. A diagram containing only gluons is a manifold without boundaries (e.g. the sphere S^2), while a diagram with a quark loop (Fig. 3) gives a manifold with a boundary (e.g. a sphere with a hole in it). Following Manohar [4], we rescale the fields to simplify the power-counting. We define: $\hat{A} = gA/\sqrt{N}$ and $\psi = \sqrt{N}\hat{\psi}$. N disappears from the covariant derivative, $D_\mu = \partial_\mu + i\hat{A}_\mu$; the entire N-dependence of the Lagrangian is now confined to an overall factor:

$$\mathcal{L} = N\left[-\frac{1}{2g^2}\mathrm{Tr}\,\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \sum_{i=1}^{N_f} \hat{\bar{\psi}}_i(i\gamma^\mu D_\mu - m_i)\hat{\psi}_i \right]. \tag{11}$$

Each vertex gives a factor of N, while each propagator—an edge of the polyhedron—gives $1/N$. Summing over the color index in a loop—a face of the polyhedron—gives an additional N. Putting these together, the overall order of a given diagram is N^{V-E+F} (where V is the number of vertices, F is the number of faces, and E is the number of edges). This particular combination $V - E + F = \chi$ is a topological invariant known as the *Euler characteristic*. For a generic connected manifold, it is defined in terms of the number of handles h and boundaries b: $\chi = 2 - 2h - b$. By examining the topology of each graph, we can determine its order. [9]

The sphere (with $h = 0$ and $b = 0$) has the greatest possible Euler characteristic: graphs with spherical topology are $\mathcal{O}(N^2)$. Such diagrams consist entirely of gluon lines which do not "jump over" each other. While a gluon line gives the boundaries of two contiguous edges in the double-line

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[6]For an $SO(N)$ gauge theory we have only one fundamental representation, because the fundamental is real. The fermions in such a theory are Majorana. This means that we would have no arrows on the propagator lines, so the double-line notation gives unoriented manifolds [4].
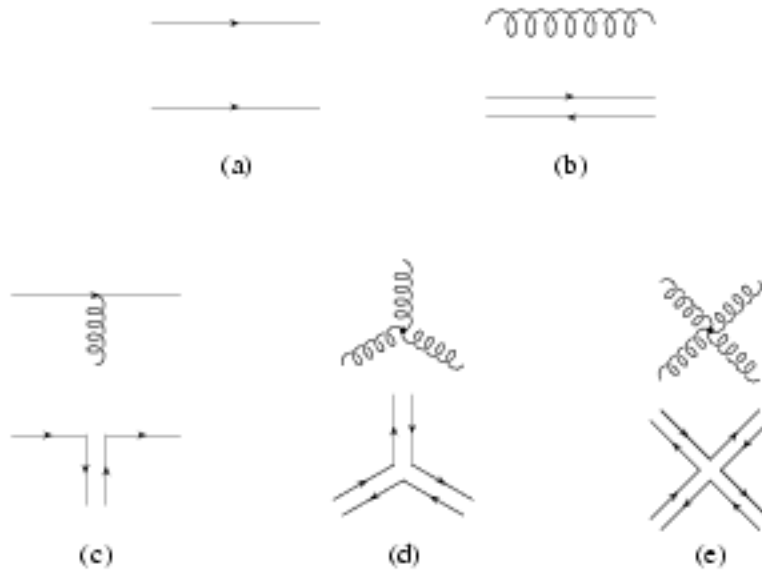
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Figure 2: Vertices and propagators of an $SU(N)$ gauge theory and their equivalents in the double-line notation. Note, in (c),(d),(e), that "color flow" has no sources or sinks.
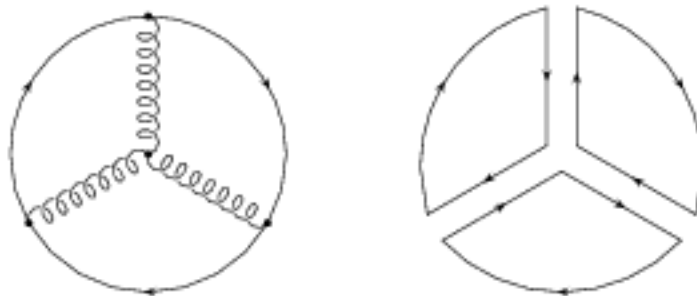

Figure 3: A planar diagram with one quark loop. It has the topology of a sphere with one boundary, and thus has Euler characteristic $\chi = 2 - 1 = 1$, giving it $\mathcal{O}(N)$.

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notation, a quark line defines an unpaired edge on the manifold; that is, each closed quark line adds a boundary. A gluon line which crosses over another line without intersecting changes the topology by adding a handle. Polyhedral diagrams in which removal of one face allows us to deform the surface to a flat plane are termed *planar diagrams*. They include vacuum diagrams consisting of gluons which can even have multiple three- or four-leg vertices on their interior, as long as the gluons do not come out of the page and the outer boundary is a full gluon loop. The magic of the $1/N$ expansion is that *all* such vacuum diagrams are $\mathcal{O}(N^2)$ – each additional flat gluon line merely adds a face to the polyhedron, leaving unchanged the fact that it can be deformed continuously into a sphere. A planar vacuum diagram which includes a single fermion loop as the outermost edge (Fig. 3) is a sphere with one boundary, as so is $\mathcal{O}(N)$. The diagram in Fig. 4 is not planar even though we can draw it on a flat sheet of paper. We can deform it to have the external gluon loop jumping over the internal one: it is a diagram of order $1/N$, because it has $h = 1$ and $b = 1$ so $\chi = -1$. (Alternatively, we can see that it has 4 vertex factors which give N^4, one color loop giving N, and 6 internal lines: it is indeed $\mathcal{O}(N^{-4/2+1}) = \mathcal{O}(1/N)$.) A diagram with a fermion loop is only truly planar if the fermion defines the external boundary.

2.3 Counting N's for Mesons and Glueballs

From the vacuum diagram in Fig. 3, we may deduce that the $1/N$ expansion generates an effective theory of mesons. A meson is a color singlet quark-antiquark pair. In the double-line notation, each meson arises as a "puncture" on the boundary of the diagram – that is, a "source term" on a quark propagator – which is the same as generating a quark-antiquark pair from the vacuum. The order of a diagram depends only on its topology and on the number of such meson vertices inserted. A planar diagram with two meson insertions gives a meson propagator with self-energy corrections from all orders in the coupling g, since adding gluon lines inside the quark line boundary does not affect the order of N. For a given topology, we will see that each additional meson insertion is suppressed by a factor of $1/\sqrt{N}$, so $1/\sqrt{N}$ thus serves as an effective coupling constant.

Let us now concretely derive the N-counting rules for more general insertions. We require (1) that these composites of quarks and gluons be gauge-invariant color singlets, (2) that all terms including quarks be quark
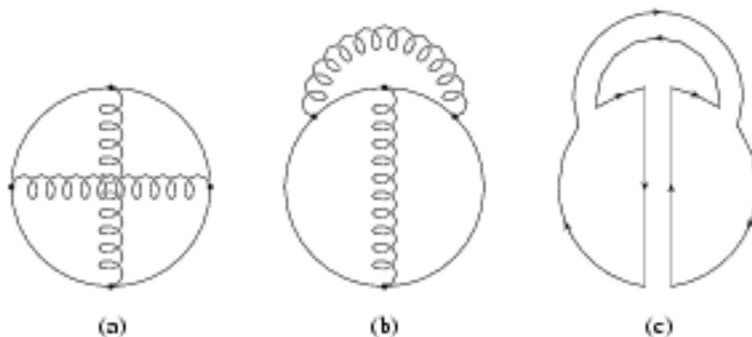
Figure 4: An example of a nonplanar diagram. (a) shows one gluon line "jumping over" the other, (b) shows an equivalent representation of the same graph, in which we see that even diagrams which can be drawn in two dimensions are only planar if all of their outer boundary is a quark loop (or all a gluon loop). (c) presents the same diagram in double-line notation: we see that the extra gluon line adds a handle to the topology.

bilinears, and (3) that the color singlet arise from a *single* color trace. The first two conditions impose the empirical observation of color confinement, while the final ones guarantees the accuracy of our N-counting. For instance, we allow operators such as $\hat{\bar{\psi}}\hat{\psi}$, but not $(\hat{\bar{\psi}}\hat{\psi})^2$, which has two separate color traces. The second operator should have twice as many N-factors as the first, but would be artificially given the same order in our counting. The simplest gluon operator would have the form $\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}$.

We can produce n-point correlators from a generating functional $W[J]$. In the usual path integral formalism [2], we insert source terms of the form $NJ_i\hat{O}_i$, where J_i is a classical field source and $\hat{O}_i$ is a color singlet operator: $\exp(W[J]) = \int \mathcal{D}\hat{\psi}\mathcal{D}\hat{A} \exp(i\int \mathcal{L} + \sum_{i=1} NJ_i\hat{O}_i)$. The factor N in the interaction term gives it the same order of N as the Lagrangian (Eq. (11)). $W[J]$ represents the sum of all connected vacuum diagrams. An n-point correlation function is given by

$$\left\langle \hat{O}_1\hat{O}_2\ldots\hat{O}_n \right\rangle = \sum_{i=1}^{n} \left(\frac{1}{iN}\frac{\delta}{\delta J_i}\right) W[J]\Big|_{J=0}. \qquad (12)$$

For operators consisting entirely of gluons, the leading terms in the correlator receive N^2 from the leading vacuum diagrams in $W[J]$, and N^n from the

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factors of $1/N$ on the right-hand-side of Eq. (12), for a total of N^{2-n}. For operators containing quark bilinears, the leading order vacuum diagrams are $\mathcal{O}(N)$, so an n-point function of quark bilinears is $\mathcal{O}(N^{1-n})$. A mixed correlation function must consist of glueball and meson insertions in a quark loop vacuum diagram. For n pure gluon operators and m quark biliners, such a correlator is $\mathcal{O}(N^{1-n-m})$.

We may now use the N-counting rules for gluons and quarks to compute the order of glueball ($\equiv \hat{G}$) and meson ($\equiv \hat{M}$) scattering amplitudes. From the counting above, we see that a pure gluon operator $\hat{G}$ creates glueballs from the vacuum with unit amplitude: $\left\langle \hat{G}_1 \hat{G}_2 \right\rangle \sim \mathcal{O}(N^{2-2}) = \mathcal{O}(1)$. Each additional glueball suppresses the amplitude by a factor $1/N$. The meson two-point function has order $\mathcal{O}(N^{1-2}) = \mathcal{O}(1/N)$. By definition the residue of the meson propagator pole must be $\mathcal{O}(1)$, and its location must give the meson mass: we take instead $\sqrt{N}\hat{M}$ to be our "physical" meson operator. Now each additional meson causes a $1/\sqrt{N}$ suppression. For a mixed correlation function of n glueballs and m mesons we have $\left\langle \hat{G}_1 \hat{G}_2 \ldots \hat{G}_n \sqrt{N}\hat{M}_1 \ldots \sqrt{N}\hat{M}_m \right\rangle \sim \mathcal{O}(N^{1-n-m/2})$. This is indeed an effective theory of mesons and glueballs interacting weakly with a coupling $1/\sqrt{N}$.

As for any field theory, we can expand the Lagrangian to various orders of the small coupling $1/\sqrt{N}$: leading order diagrams are tree-level graphs such as the one shown in Fig. 3 with the appropriate number of insertions on the boundary. Adding a gluon loop contributes a handle, and thus a factor of $1/N$. The $\mathcal{O}(N^2)$ and $\mathcal{O}(N)$ terms reflect the confinement of quarks and gluons into color singlets [4], while all further interactions are suppressed by factors of $1/N$. We now deduce that the $1/N$ expansion gives a semiclassical limit of the effective meson theory – we say "semiclassical" because if we restore units in the path integral, we find that we are actually expanding in $\hbar/N$, which we know is also a loop expansion.

The full spectrum includes an infinite tower of meson and glueball resonances, which are necessary to give the appropriate logarithmic running of QCD correlators. Each resonance has a narrow decay width, since all vertices leading to decays are proportional to the coupling $1/\sqrt{N}$. Indeed, we can now write any meson two-point function as a sum of resonances [4]:

$$\int d^4x e^{ipx} \left\langle \bar{q}(x)q(0) \right\rangle = \sum_k \frac{Z_k}{p^2 - m_k^2}, \qquad (13)$$
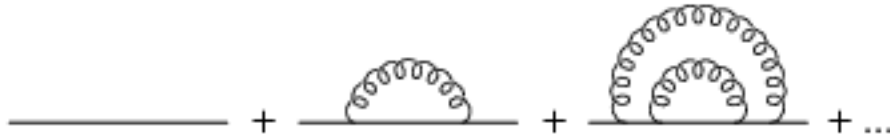
Figure 5: "Rainbow" diagrams [4] determine the self-energy corrections to the fermion propagator in the 't Hooft model. Planarity and absence of gluon-gluon interactions guarantee that these are the only 1PI contributions

where the m_k are the masses of the meson resonances, and the residues Z_k are weights associated with each resonance.

3   Applications and Extensions

3.1   The 't Hooft Model

While these results are interesting, they are not easy to apply. QCD in (3+1) dimensions still has no closed form expansion in $1/N$. A simpler model developed by 't Hooft [10] serves to clarify the abstract notions presented so far. It *is* exactly solvable in the large N expansion, taking $N \to \infty$ while keeping the so-called *'t Hooft coupling*, g^2/N, fixed).

We consider N-color QCD in 1+1 dimensions with coupling g. The model displays confinement in the form of a linear potential increasing at large distances. Rewriting the standard Lagrangian in light-cone coordinates ($x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$) [7] simplifies the problem greatly, as making the gauge choice ($A^+ = A_- = 0$) leaves us with a field strength consisting of a single component $F_{+-} = \partial_+ A_- - \partial_- A_+ + (ig/\sqrt{N})[A_+, A_-] = -\partial_- A_+$. Eliminating the commutator of the gluon fields in the field strength renders the theory essentially abelian; the only gluon vertices are those with one gluon, one quark, and one antiquark. We no longer have three- and four-gluon vertices. Since we send $N \to \infty$, all contributing diagrams are planar: this forbids gluon lines jumping over each other, so the allowed diagrams are almost trivial. Because of these simplifications, we can calculate the quark propagator exactly to leading order in $1/N$. The only possible graphs are the

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[7]With metric of form $g_{++} = g_{--} = 0$ and $g_{+-} = g_{-+} = 1$.

"rainbows"[10] shown in Fig. 5. We acquire the noncompact quark propagator by taking the radius of the fermion loop boundary of the $\mathcal{O}(N)$ vacuum diagram to infinity. A straightforward calculation for the self-energy gives $\Sigma = -g^2/\pi p_-$, and a quark propagator of the usual form $1/(\not{p} - M)$ [2]. M is the renormalized quark mass, $M^2 = m^2 + p_+ \Sigma = m^2 - g^2/\pi$.[8] The location of the poles in the meson propagator determine the mass spectrum. The simplified Feynman rules yield a propagator given by the sum shown in Fig. 6. Applying the Bethe-Salpeter equation for bound states gives an expression for the meson wavefunction [10]:

$$\mu^2 \phi(x) = \left[\frac{M^2}{x} + \frac{M^2}{1-x} \right] \phi(x) - \frac{g^2}{2\pi} \int_0^1 dy P \left[\frac{1}{(x-y)^2} \right] \phi(y), \qquad (14)$$

where μ is the meson mass, and x is the light cone momentum fraction (described below). Let us see how this equation arises. The spectrum should essentially be given by a light cone time-independent Schrödinger equation, with eigenvalues μ^2 and eigenstates ϕ. In the lightcone formalism, x^+ is the "time" variable and x^- is the "position." The operators P_+ and P_- are the generators of x^+ and x^- translations, respectively. As Coleman [2] suggests, for two-particle systems in light cone coordinates, we can work in an eigenspace of P_- with eigenvalue 1; this means that if one particle has operator P_- given by x, the other has $1-x$ with $x \in [0,1]$. In light cone coordinates, energy-momentum conservation relates P_- and P_+ as $P_+ = M^2/(2P_-)$, so the total energy for the two-particle system is given by:

$$\mu'^2 = 2P_+ P_- = 2P_+ = \frac{M^2}{x} + \frac{M^2}{1-x}. \qquad (15)$$

We must add in by hand a potential of form $g^2|p|$, which is just the Fourier transform of the final term in Eq. (14): this is what gives us the confining potential! This potential guarantees that the wavefunction ϕ represents a bound quark pair; because the potential is infinite, the spectrum is indeed discrete. Eq. (14) can then be solved to determine the meson spectrum, as in [10]. The 't Hooft model in the large N limit exhibits many of the salient characteristics of 3+1-dimensional QCD, such as the exclusion of free quarks, and may give hints for tackling the more difficult (3+1)-dimensional problem.

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[8]Note that we must ignore the small values of m for which this physical quark mass is tachyonic. These are artifacts of the theory; the meson mass spectrum is otherwise sensible.
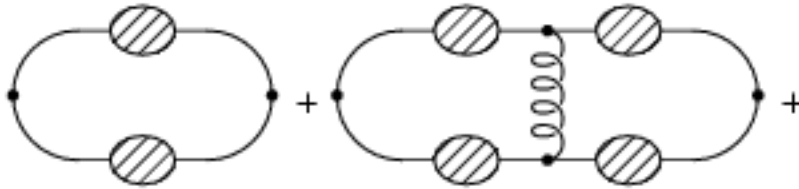
Figure 6: The meson two-point function in the 't Hooft model. The shaded circles refer to the full, corrected quark propagator. Again, these are the only possible contributing diagrams which preserve planarity and obey the $1 + 1$-dimensional Feynman rules.

3.2   Meson Phenomenology

Many salient empirical features of mesons arise naturally in the $1/N$ expansion of QCD. The simplest among these is *Zweig's rule*, the statement that all processes in which a constituent quark of some hadron begins and ends in the same hadron are suppressed [11]. The corresponding diagram (Fig. 9) has two quark loops, and so is indeed suppressed by a factor of $1/N$ compared to the one-quark-loop diagrams.

Exotics—bound states consisting of four or more quarks—are also not present to leading order in $1/N$ [2]. A gauge-invariant quark quadrilinear, for instance, would have a leading term in its propagator just corresponding to the propagation of two separate meson states (which, as discussed above, is $\mathcal{O}(1)$. A term consisting of a bound state of two mesons would be $\mathcal{O}(N^{1-2}) = \mathcal{O}(1/N)$, and does not appear to leading order. (It can arise, of course, at subsequent orders in $1/N$.)

Further results of the large N meson phenomenology, which we will not discuss in detail, include the realization of chiral symmetry-breaking, which generates the pions as pseudo-Goldstone bosons. The low-energy effective theory may be described in terms of the so-called chiral Lagrangian, which results from integrating out the fields of the usual QCD partition function with source terms for the pseudo-Goldstones [12]. These source terms are quark bilinears. As we have seen in our discussion of meson N-counting rules, the correlators of bilinears are given by single-fermion-loop diagrams (to leading order). The leading order terms in the effective Lagrangian are therefore order N. Each additional meson in a correlator brings an extra factor of $1/\sqrt{N}$, so interactions of three or more mesons are suitably suppressed. A diagrammatic expansion yields a Lagrangian corrected to various orders of N, with coefficients found to be in agreement with those determined

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experimentally [7].

3.3 Baryons

Given the successes of the large N expansion for reproducing light meson phenomena, it is natural to wonder if the large N expansion can offer a consistent description of baryons. In 1979, Witten [12] proposed a sensible expansion even for baryons.

Baryons are color singlets consisting of N quarks antisymmetric in color space: $\epsilon_{i_1 \ldots i_N} q^{i_1} \ldots q^{i_N}$. Since they are fermionic objects, the isospin and spatial wavefunctions together must be symmetric − that is, outside of color space, baryons look like bosons.[9] For large-N purposes, baryons are a bundle of N quark lines, each of different color, propagating together. A naive analysis suggests that our usual expansion is impossible: the self-energy corrections to the baryon propagator appear to have *increasing* powers of N. Consider a correction to the baryon self-energy consisting of a gluon exchange between two of the constituent quarks − a "two-body interaction." Each quark-gluon vertex contributes a factor of $1/\sqrt{N}$, for a total of $1/N$. However, there are $N(N-1)/2$ such quark pairings possible, so the self-energy receives a correction of order $N^2/N = N$. Similarly, an exchange of two gluons gives a contribution of order N^2, and so on.

The solution lies in the value of the baryon mass [12]. If we assume that the baryon is very heavy, the sum over all diagrams of the type described above should yield an quantum-mechanical propagator:

$$e^{-iM_B t} = 1 - iM_B t - \frac{M_B^2 t^2}{2} + \ldots \qquad (16)$$

$$= 1 + \mathcal{O}(N) + \mathcal{O}(N^2) + \ldots, \qquad (17)$$

where in the second line we have recorded the self-energy contributions from higher and higher numbers of gluon exchanges. The apparent divergence need not cause alarm: in fact, we have just found that the baryon mass M_B is $\mathcal{O}(N)$.

Though the double-line diagrammatics used for mesons and glueballs are less transparent here, they are still useful for deriving the N-counting rules for baryons. As noted, we think of each baryon as a bundle of N quark

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[9]This was, in fact, the reason that the color quantum number was proposed in the first place.
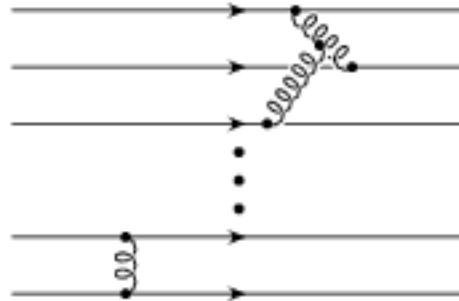
Figure 7: Disconnected two-body interactions giving corrections to the baryon propagator. Each connected interaction is treated separately: the product of the orders of the connected interactions give the order of the diagram, in this case, $\mathcal{O}(N \cdot N) = \mathcal{O}(N^2)$.

lines, one of each color, propagating together.[10] An interaction between the quarks simply permutes indices on the lines. We call an "n-body interaction" one which mixes the indices of n of the constituent quarks – in such a way that this mixing is connected (it cannot be reduced to two or more smaller sets permuting among themselves). Disconnected interactions, such as that shown in Fig. 7, contribute at the order which is the *product* of the constituent connected interactions. To leading order, these n-body interactions correspond to the connected single-quark-loop vacuum diagrams we analyzed above. The n quarks appear as n punctures in the boundary loop – except that now we must "twist"[4] one of the quark lines to have all of the quarks propagating in the same direction. Fig. 8 shows that though the diagram shown appears nonplanar (we have a gluon line jumping over another gluon line), the discrepancy arises from the twist. If we flip the bottom quark line in Fig. 8, we find a conventional planar loop diagram, where the external quark boundary stretched out to infinity, so that it looks like two separate constituent quark lines. Therefore, the topology of the vacuum loop which gives us the proper power of N has not changed.

Note that interactions of all n values are equally important. From the vacuum loop, we see that an n-body interaction is $\mathcal{O}(N^{1-n})$ since each loop we cut through and label with a definite index takes away a color index sum, and thus a power of N. However, there are $\mathcal{O}(N^n)$ ways of picking out

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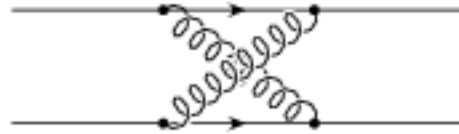[10]We neglect flavor for the moment.

Figure 8: A two-body interaction which appears nonplanar, though it actually is. Note that rotating the bottom line by π about an axis lying in the page, but perpendicular to the quark line gives a planar vacuum diagram, since the two gluon lines are now parallel.

which of the N quarks participate in the interaction: overall, any n-body interaction is $\mathcal{O}(N)$ – independent of n.

The N-counting rules and the quantum mechanical, many-body description of the baryon also extend to interactions between baryons and other color singlet operators. For instance, the process $B + \bar{q}q \rightarrow B$ (where the baryon absorbs some hypothetical meson) corresponds to the matrix element $\langle B|\bar{q}q|B \rangle$. The meson operator can be placed on any of the N quark lines in the baryon, so this process is at most $\mathcal{O}(N)$. (If cancellations occur among the N contributions, the order may be lower.) Generally, an n-body operator produces an amplitude of at most $\mathcal{O}(N^n)$ [4].

We can deduce the meson-baryon coupling from this matrix element, which is $\mathcal{O}(N)$. The amplitude for a fermion bilinear to create a meson is $\sqrt{N}$. Note that we must consider *physical* mesons to find the actual order of the coupling. We found that such bilinears are scaled to $\sqrt{N}\hat{M}$ to give $\mathcal{O}(1)$ meson propagators. The fermions in the bilinear, $\hat{\psi}$, have been rescaled from the physical quarks in the original Langrangian: $\psi = \sqrt{N}\hat{\psi}$. In terms of physical quarks, then, the expectation value $\left\langle B|\sqrt{N}M|B \right\rangle = \sqrt{N}\left\langle B|\bar{\hat{\psi}}\hat{\psi}|B \right\rangle = \langle B|\bar{\psi}\psi|B \rangle/\sqrt{N}$. The coupling between the meson and baryon, then, must be at most $\mathcal{O}(\sqrt{N})$. Say we add another meson, to give meson-baryon scattering. For conservation of energy, both mesons must be inserted on the same quark line (at leading order in $1/N$) – this gives a multiplicity of N. Each meson, as described, comes with a factor of $1/\sqrt{N}$ – so overall the scattering amplitude is $\mathcal{O}(1)$. As was mentioned in the case of glueball-meson and pure mesonic amplitudes, each additional mesons further suppresses the process by $1/\sqrt{N}$.

Finally, we can consider baryon-baryon scattering at fixed velocities. The

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baryons are extremely heavy, so they basically have a classical momentum $p = M_B v$. The scattering process will involve the interaction of at least one quark from each N-quark baryonic bundle. The choice of quarks within the bundle contributes a combinatoric factor of N^2. To couple quarks of different colors we need a gluon, which contributes $1/\sqrt{N}^2$. (If no gluon is exchanged, the combinatoric factor is just N: either way we find an $\mathcal{O}(N)$ process.) Since the amplitude is $\mathcal{O}(N)$, and the baryon mass is $\mathcal{O}(N)$, baryons essentially exhibit classical scattering as N grows large [12].

Though we will not discuss this interesting point at length here, we note that the large N expansion has yielded an effective theory in which mesons are fundamental (unit mass) fields with a small coupling $g_M \equiv 1/\sqrt{N}$. The baryon masses go as inverse powers of the coupling, which suggests that they might appear as solitonic objects. Such an effective meson theory was proposed by Skyrme [8] and developed further by Adkins, Nappi, and Witten [1], who indeed posited that baryons arise as solitons of the pion fields in the Skyrme model.

3.3.1 The nonrelativistic quark model

To define the morphology of the baryon spectrum, we think of the baryon as a nonrelativistic bound state of heavy quarks. The Hamiltonian of the system is [12]

$$ H = NM + \sum_i \frac{p_i^2}{2m} + \frac{1}{N} \sum_{i \neq j} V(x_i - x_j) + \frac{1}{N^2} \sum_{i \neq j \neq k} V(x_i - x_j - x_k) + \dots \quad (18) $$

where each of the x_i denotes the position of a quark, and M is the quark mass. By the counting conducted above, we see that each term is $\mathcal{O}(N)$. We assume that all of the quarks move in an average background potential [12]. The groundstate wavefunction of the baryon is $\psi_0(x_1 \dots x_N) = \prod_{i=1}^{N} \phi_0(x_i)$, with ϕ_0 the groundstate of an individual quark in the background potential.

Let us first review the non-relativistic quark model for QCD, with three light quark flavors (u, d, s) and $N = 3$. The spin, flavor and spatial wavefunctions together must be *symmetric* since the color wavefunction is antisymmetric. Each of the three flavor quarks can have spin $\pm 1/2$, the groundstate of the baryon must transform as a completely symmetrized product of three fundamentals of $SU(6)$ (since $6 = 2 \cdot 3$, with 2 from spin and 3 from the

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approximate 3-flavor symmetry). The three spins can add up to either $3/2$ or $1/2$. The $j_z = 3/2$ spin wavefunction is symmetric (all spin up), so the flavors must also exhibit a symmetric state: for instance, ddd, or sss, or $(uud + udu + duu)/\sqrt{3}$, etc. These constitute the "decuplet baryons" (the Δ's, Σ^*'s, Ξ's and Ω^-) [4]. Empirical observation dictates that spin $1/2$ baryons having all three quarks of the same flavor do not arise in nature. Let us then consider baryons, like the proton, in which only two quarks are identical. The two identical flavor quarks must have a totally symmetric spin wavefunction. For $j = 1/2$, the wavefunction of the proton is

$$\frac{uud}{\sqrt{6}} [2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow] + \text{cyclic permutations} \qquad (19)$$

where the cyclic permutations guarantee that the flavor + spin wavefunction is completely symmetric. This gives six of the octet baryons. The remaining ones have all three quarks different, such as the Σ^0 and the Λ.

The defining properties of the baryons arise naturally in this model [4]. The net magnetic moment is given by $\mu_f \sigma^3$, where μ_f is the magnetic moment of a given flavor. For instance, the proton has magnetic moment $4\mu_u/3 - \mu_d/3 = -3\mu_d$.

Another parameter, the axial coupling, is given relativistically by $\langle p | \bar{q} \gamma^\mu \gamma_5 \tau^3 q | p \rangle$ (where τ acts on the flavor $SU(3)$). The nonrelativistic version of this is the expectation value $q^\dagger \sigma^3 \tau^3 q$, which give $+1$ for $d \downarrow$ and $u \uparrow$, and -1 for $u \downarrow$ and $d \uparrow$. A simple calculation [4] determines that the proton, uud, has $g_a = 5/3$. We derive this be applying the axial current given above to the proton wavefunction.

Now consider the equivalent analysis for N colors. We choose $N = 2m+1$ (m integer) and again take three flavors. Requiring total antisymmetry of the color wavefunction yields a symmetric total spin + flavor + space wavefunction. We assume that the quarks are all in the spatial wavefunction groundstate, so it remains only to construct a symmetric flavor + spin wavefunction, which is an N-index symmetric tensor of $SU(6)$. Under $SU(2)_{spin} \times SU(3)_{flavor}$, the tensor of $SU(6)$ decomposes into representations labeled by the J-value with $J = 1/2, J = 3/2, \ldots J = N/2$. Let us focus specifically on the $1/N$ proton, which we define to have spin and isospin $1/2$ [5]. This implies that it has $(m + 1)$ u's and m d's, each of which must form symmetric spin wavefunctions among themselves. As a result, the u's contribute a total of $J = (m + 1)/2$ and the d's $J = m/2$ – together, they must form total $J = 1/2$. Using these results we can compute g_A as above,

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by applying $\sigma^3 \tau^3 = 2(J_{(u)}^3 - J_{(d)}^3)$. The Wigner-Eckart theorem and some calculation[5] give $g_A = (N + 2)/3$, which indeed reduces to 5/3 for $N = 3$. The axial coupling is thus $\mathcal{O}(N)$ – this is the result we would expect from our large N counting rules for the matrix element in the baryon of single-quark operator.

4 Conclusion

While 't Hooft's large N expansion is used for phenomenological prediction with some hesitation, it is nonetheless an important tool for studying QCD as well as applications in statistical physics and the more esoteric AdS/CFT correspondence. The expansion is so important because it is fundamentally nonperturbative. We expand around a different vacuum from the usual zero-coupling, free theory vacuum: because our expansion parameter is $1/N$ instead of the coupling, g, we can uncover much valuable information about the structure of strongly coupled physics of all types.

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